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Asymptotic Formulae for a Particular Solution of Linear Nonhomogeneous Discrete Equations

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Abstract—The aim of this contribution is to develop the asymptotic formulae which describe the asymptotic behaviour of a particular solution of linear nonhomogeneous differential equation $\Delta u(k) = A(k)u(k) + g(k)$ if $k \to \infty$. It is shown that asymptotic formulae are generated by partial sums of certain formal series that satisfy the given equation. A comparison with the known explicit formula for solution of the initial problem is given and, moreover, some illustrative examples are discussed. The used method of investigation is based on results published in [1].

Keywords—Linear discrete equation, Particular solution, Asymptotic formulae.

1. INTRODUCTION

In this paper, we consider the linear discrete nonhomogeneous equation

$$\Delta u(k) = A(k)u(k) + g(k), \qquad k \in N(a), \tag{1}$$

where $\Delta u(k) = u(k+1) - u(k)$, $N(a) = \{a, a+1, \dots\}$, $a \in \mathbb{N}$ is fixed, $\mathbb{N} = \{0, 1, \dots\}$, and $A, g : N(a) \to \mathbb{R}$. We present asymptotic formulae describing the asymptotic behaviour of a particular solution of equation (1). A formal series satisfying (1) is constructed. We show that under certain conditions, its partial sums describe asymptotic behaviour of a particular solution of equation (1) in the case when a is sufficiently large. The proofs of corresponding results use the results in [1] concerning asymptotic behaviour of a solution of a general nonlinear discrete equation. The results obtained are illustrated on examples. Except for this, a comparison with the known explicit formula, describing the solution of an initial problem for equation (1), is given. Problems of the asymptotic behaviour of solutions of difference equations are always in the centre of interest and various aspects of it were studied, e.g., in [2–10].

2. PRELIMINARIES

Let us consider the scalar difference equation

$$\Delta u(k) = f(k, u(k)), \tag{2}$$

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where $f(k,u)$ is defined on $N(a) \times \mathbb{R}$ with values in $\mathbb{R}$ and with $N(a)$ defined as above. The existence and uniqueness of the solution of the initial problem (2),(3) where

$$u(a+s) = u^s \in \mathbb{R}^n, \qquad s \in \mathbb{N}, \quad (s \text{ is fixed}), \tag{3}$$

on $N(a+s)$ is obvious [11, p. 6]. Let us recall that the solution of the initial problem (2),(3) is defined as an infinite sequence of numbers

$$\{u(a+s), u(a+s+1), u(a+s+2), \dots, u(a+s+n), \dots\},$$

with $u(a+s) = u^s$ such that for any $k \in N(a+s)$, equality (2) holds.

We suppose that for all $(k,u) \in N(a) \times \mathbb{R}$ and $(k,v) \in N(a) \times \mathbb{R}$,

$$|f(k,u) - f(k,v)| \le \lambda(k)|u - v|,$$

where $\lambda(k)$ is a nonnegative function defined on $N(a)$. Then the initial problem (2),(3) depends continuously on initial data (see, e.g., [11]).

Let us define sets

$$\omega = \{(k,u) : k \in N(a),\ u \in \mathbb{R},\ b(k) < u < c(k)\},$$

where $b(k)$, $c(k)$, $b(k) < c(k)$ are real functions defined on $N(a)$ and

$$\omega(k) = \{u : u \in \mathbb{R},\ b(k) < u < c(k)\}.$$

THEOREM 1. *(See [1].) Let us suppose that $f(k,u)$ is defined on $N(a) \times \mathbb{R}$ and for all $(k,u) \in N(a) \times \mathbb{R}$ and $(k,v) \in N(a) \times \mathbb{R}$,*

$$|f(k,u) - f(k,v)| \le \lambda(k)|u - v|, \tag{4}$$

where $\lambda(k)$ is a nonnegative function defined on $N(a)$. Let there exist real functions $b(k)$, $c(k)$, $b(k) < c(k)$, defined on $N(a)$ such that the inequalities

$$f(k, b(k)) - b(k+1) + b(k) < 0,$$
$$f(k, c(k)) - c(k+1) + c(k) > 0$$

hold for every $k \in N(a)$. Then there exists an initial problem for equation (2):

$$u(a) = u^*,$$

with $u^* \in \omega(a)$ such that the corresponding solution $u = u^*(k)$ of equation (2) satisfies the relation $(k, u^*(k)) \in \omega$ on $N(a)$, i.e.,

$$b(k) < u^*(k) < c(k),$$

for every $k \in N(a)$.

This theorem is used in the following to investigate linear equation (1). Therefore, we reformulate it as a corollary. The validity of Lipschitz type condition (4) for linear equation (1) is obvious.

COROLLARY 1. *Suppose the existence of real functions $b(k)$, $c(k)$ with $b(k) < c(k)$, defined on $N(a)$. Let the inequalities*

$$A(k)b(k) + g(k) - b(k+1) + b(k) < 0, \tag{5}$$
$$A(k)c(k) + g(k) - c(k+1) + c(k) > 0 \tag{6}$$

hold for every $k \in N(a)$. Then there exists an initial problem for equation (1):

$$u(a) = u^*,$$

with

$$b(a) < u^* < c(a)$$

such that the corresponding solution $u = u^*(k)$ of equation (1) satisfies the inequalities

$$b(k) < u^*(k) < c(k),$$

for every $k \in N(a)$.

3. ASYMPTOTIC FORMULAE

Consider equation (1), i.e., the equation

$$\Delta u(k) = A(k)u(k) + g(k), \qquad k \in N(a),$$

and suppose $A(k) \neq 0$ for every $k \in N(a)$. Let us construct a formal series which satisfies equation (1). Define a sequence of functions

$$f_0(k), f_1(k), \dots, f_n(k), \dots, \qquad k \in N(a), \tag{7}$$

as follows

$$f_0(k) = -\frac{g(k)}{A(k)}, \qquad k \in N(a), \tag{8}$$

and

$$f_p(k) = \frac{\Delta f_{p-1}(k)}{A(k)}, \qquad k \in N(a), \tag{9}$$

where $p = 1, 2, \dots$. Obviously, the sequence (7) is, in view of formulas (8),(9), well defined for every $k \in N(a)$. Define a *formal series*

$$\mathcal{FS}(k) := f_0(k) + f_1(k) + \cdots + f_n(k) + \cdots. \tag{10}$$

LEMMA 1. *Suppose $A(k) \neq 0$ for every $k \in N(a)$. Then the formal series $\mathcal{FS}(k)$ defined by relation (10) is a formal solution of equation (1).*

PROOF. This is easy to verify. By putting the expression $u(k) \equiv \mathcal{FS}(k)$ into equation (1), we see that this equation is satisfied.

REMARK 1. The formal series (10) equals, in the case of a concrete equation of type (1), a divergent series or to a convergent series. In the latter case, this series represents a particular solution of equation (1). However, in the general case it can be a nontrivial matter to establish the divergence or the convergence of the formal series (10). Let us illustrate both possibilities on appropriate equations.

EXAMPLE 1. Consider an equation of type (1) with $A(k) = 1/k$ and $g(k) = -1$, i.e., the equation

$$\Delta u(k) = \frac{1}{k}u(k) - 1, \qquad k \in N(1), \tag{11}$$

having a divergent formal series. Indeed, it is easy to see that the formal series for (11)

$$\mathcal{FS}(k) := k + k + \cdots + k + \cdots,$$

constructed in accordance with formulae (8)–(10), is divergent for every $k \in N(1)$.

EXAMPLE 2. Consider an equation of type (1) with $A(k) = 1$ and $g(k) = -k$, i.e., the equation

$$\Delta u(k) = u(k) - k, \qquad k \in N(a), \tag{12}$$

having a convergent formal series. Indeed, it is easy to see that the formal series for (12)

$$\mathcal{FS}(k) := k + 1 + 0 + 0 + \cdots + 0 + \cdots,$$

constructed in accordance with formulae (8)–(10), is convergent for every $k \in N(a)$ and $a \in \mathbb{N}$ and defines a particular solution of equation (12).

In the following, sufficient conditions for the characterization of at least one particular solution $u^{\text{part}} = u^{\text{part}}(k)$, $k \in N(a)$, of the discrete linear nonhomogeneous equation (1) by means of partial sums of formal series $\mathcal{FS}(k)$ (independently on convergence or divergence of formal series $\mathcal{FS}(k)$ itself) are given.

THEOREM 2. *Let us suppose that for every $k \in N(a)$ and a fixed $p \in \{0\} \cup \mathbb{N}$,*

(1) $A(k) \neq 0$;

(2) $f_{p+1}(k) < 0$, $\Delta f_p(k) < 0$, *and* $\Delta f_{p+1}(k) > 0$.

Then there exists a particular solution $u^{\text{part}} = u^{\text{part}}(k)$, $k \in N(a)$, of the discrete linear nonhomogeneous equation (1) such that the inequalities

$$\sum_{s=0}^{p+1} f_s(k) < u^{\text{part}}(k) < \sum_{s=0}^{p} f_s(k) \tag{13}$$

hold for every $k \in N(a)$.

PROOF. Let us transform $u(k)$ in (1) by the formula

$$u(k) = \sum_{s=0}^{p-1} f_s(k) + w(k), \tag{14}$$

with a new unknown variable $w(k)$. If $p = 0$, then put $\sum_{s=0}^{-1} f_s(k) = 0$. This substitution leads to the new equation

$$\Delta w(k) = A(k)w(k) - \Delta f_{p-1}(k), \qquad k \in N(a), \tag{15}$$

where, in the case $p = 0$, we put $\Delta f_{-1}(k) = -g(k)$. Let us apply Corollary 1 to equation (15). For this, put

$$\begin{aligned} b(k) &= f_p(k) + f_{p+1}(k), \\ c(k) &= f_p(k), \\ g(k) &= -\Delta f_{p-1}(k), \\ u(k) &\equiv w(k). \end{aligned}$$

Then, for every $k \in N(a)$,

$$c(k) - b(k) = f_p(k) - (f_p(k) + f_{p+1}(k) = -f_{p+1}(k) > 0.$$

Further, for every $k \in N(a)$, we get

$$\begin{aligned} A(k)b(k) + g(k) - b(k+1) + b(k) &= A(k)f_p(k) + A(k)f_{p+1}(k) \\ &\quad - \Delta f_{p-1}(k) - f_p(k+1) - f_{p+1}(k+1) + f_p(k) + f_{p+1}(k) \\ &= (A(k)f_p(k) - \Delta f_{p-1}(k)) + (A(k)f_{p+1}(k) - \Delta f_p(k)) \\ &\quad - f_{p+1}(k+1) + f_{p+1}(k) \\ &= -\Delta f_{p+1}(k) < 0 \end{aligned}$$

and

$$\begin{aligned} A(k)c(k) + g(k) - c(k+1) + c(k) &= A(k)f_p(k) - \Delta f_{p-1}(k) - f_p(k+1) + f_p(k) \\ &= (A(k)f_p(k) - \Delta f_{p-1}(k)) - \Delta f_p(k) \\ &= -\Delta f_p(k) > 0. \end{aligned}$$

So, inequalities (5),(6) hold. The conclusion of Theorem 2 is now, in view of transformation (14), a straightforward consequence of Corollary 1. This completes the proof. ∎

THEOREM 3. *Let us suppose that for every $k \in N(a)$ and a fixed $p \in \{0\} \cup \mathbb{N}$:*

(1) $A(k) \neq 0$;

(2) $f_{p+1}(k) > 0$, $\Delta f_p(k) > 0$, *and* $\Delta f_{p+1}(k) < 0$.

Then there exists a particular solution $u^{\text{part}} = u^{\text{part}}(k)$, $k \in N(a)$, of the discrete linear nonhomogeneous equation (1) such that the inequalities

$$\sum_{s=0}^{p} f_s(k) < u^{\text{part}}(k) < \sum_{s=0}^{p+1} f_s(k) \tag{16}$$

hold for every $k \in N(a)$.

PROOF. Let us, as in the proof of Theorem 2 above, transform $u(k)$ in (1) by formula (14) with a new unknown variable $w(k)$ and consider a new equation (15). Apply Corollary 1 again in the case when

$$\begin{aligned} b(k) &= f_p(k), \\ c(k) &= f_p(k) + f_{p+1}(k), \\ g(k) &= -\Delta f_{p-1}(k), \\ u(k) &\equiv w(k). \end{aligned}$$

Then for every $k \in N(a)$,

$$c(k) - b(k) = (f_p(k) + f_{p+1}(k)) - f_p(k) = f_{p+1}(k) > 0.$$

Further, for every $k \in N(a)$, we get

$$\begin{aligned} A(k)b(k) + g(k) - b(k+1) + b(k) &= A(k)f_p(k) - \Delta f_{p-1}(k) - f_p(k+1) + f_p(k) \\ &= (A(k)f_p(k) - \Delta f_{p-1}(k)) - \Delta f_p(k) = -\Delta f_p(k) < 0 \end{aligned}$$

and

$$\begin{aligned} A(k)c(k) + g(k) - c(k+1) + c(k) &= A(k)f_p(k) + A(k)f_{p+1}(k) \\ &\quad - \Delta f_{p-1}(k) - f_p(k+1) - f_{p+1}(k+1) + f_p(k) + f_{p+1}(k) \\ &= (A(k)f_p(k) - \Delta f_{p-1}(k)) + (A(k)f_{p+1}(k) - \Delta f_p(k)) \\ &\quad - \Delta f_{p+1}(k) \\ &= -\Delta f_{p+1}(k) > 0. \end{aligned}$$

So, inequalities (5),(6) hold again. The conclusion of Theorem 3 is now, in view of transformation (14), a direct consequence of Corollary 1. The proof is complete. ∎

The following corollary is a consequence of Theorems 2 and 3.

COROLLARY 2. ASYMPTOTIC FORMULA FOR PARTICULAR SOLUTION. *Let Theorem 2 or 3 hold. If, moreover,*

$$\lim_{k\to\infty} \frac{f_{p+1}(k)}{f_p(k)} = 0,$$

then for the corresponding particular solution $u^{\text{part}}(k)$ of equation (1), satisfying the inequalities (13) or (16), the asymptotic relation

$$u^{\text{part}}(k) = \sum_{s=0}^{p-1} f_s(k) + f_p(k)(1 + o(1)) \tag{17}$$

holds if $k \to \infty$ where the symbol o is Landau order symbol.

4. EXAMPLES

EXAMPLE 3. Let us consider a linear discrete equation

$$\Delta u(k) = ku(k) - 1. \tag{18}$$

Straightforward computation (put $k > 0$) yields

$$\begin{aligned} f_0(k) &= \frac{1}{k}, \\ \Delta f_0(k) &= -\frac{1}{k(k+1)} < 0, \\ f_1(k) &= -\frac{1}{k^2(k+1)} < 0, \\ \Delta f_1(k) &= \frac{3k+2}{k^2(k+1)^2(k+2)} > 0. \end{aligned}$$

In accordance with Theorem 2 ($p = 0$), there exists a particular solution $u^{\text{part}} = u^{\text{part}}(k)$, $k \in N(1)$, of equation (18) such that the inequalities

$$\frac{1}{k} - \frac{1}{k^2(k+1)} < u^{\text{part}}(k) < \frac{1}{k}$$

hold for every $k \in N(1)$. By Corollary 2,

$$u^{\text{part}}(k) = (1 + o(1)) \cdot \frac{1}{k}, \qquad k \to \infty.$$

EXAMPLE 4. Let us consider a linear discrete equation

$$\Delta u(k) = k^5 u(k) - k^6. \tag{19}$$

Straightforward computation (put $k > 0$) yields

$$\begin{aligned} f_0(k) &= k, \\ \Delta f_0(k) &= 1 > 0, \\ f_1(k) &= \frac{1}{k^5} > 0, \\ \Delta f_1(k) &= \frac{1}{(k+1)^5} - \frac{1}{k^5} < 0. \end{aligned}$$

In accordance with Theorem 3 ($p = 0$), there exists a particular solution $u^{\text{part}} = u^{\text{part}}(k)$, $k \in N(1)$, of equation (19) such that the inequalities

$$k < u^{\text{part}}(k) < k + \frac{1}{k^5}$$

hold for every $k \in N(1)$. By Corollary 2,

$$u^{\text{part}}(k) = k \cdot (1 + o(1)), \qquad k \to \infty.$$

5. CONCLUDING REMARKS

It is easy, using mathematical induction (see, e.g., [12]), to prove that the solution of the initial problem

$$
\begin{aligned}
y(k+1) &= \alpha(k)y(k)+\beta(k), \qquad k \in N(a),\\
y(a) &= y_a
\end{aligned}
\tag{20}
$$

can be written as

$$
y(k)=\left[\prod_{s=a}^{k-1}\alpha(k)\right]y_a+\sum_{r=a}^{k-1}\left[\prod_{s=r+1}^{k-1}\alpha(s)\right]\beta(r), \qquad k\in N(a). \tag{21}
$$

In (21) it is put, by definition, $\prod_{s=k+1}^{k} a(s)=1$ and $\sum_{s=k+1}^{k} a(s)=0$. Let us rewrite the linear nonhomogeneous equation (1) in the form of equation (20), i.e., in the form

$$
u(k+1)=(1+A(k))u(k)+g(k).
$$

Then the solution of initial problem $u(a)=u_a$ can be written (put $\alpha(k)\equiv 1+A(k)$, $\beta(k)\equiv g(k)$) as

$$
u(k)=\left[\prod_{s=a}^{k-1}(1+A(k))\right]u_a+\sum_{r=a}^{k-1}\left[\prod_{s=r+1}^{k-1}(1+A(s))\right]g(r), \qquad k\in N(a). \tag{22}
$$

Formula (22) is unfortunately too cumbersome. It is not clear how to choose a suitable initial value u_a (or how to simplify the formula (22)) in order to get (in the case of validity of Theorems 2 or 3) inequalities (13),(16) or (in the case of validity of Corollary 2) relation (17) for the solution $u(k)$.

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